THE THEORETICAL TREATMENT OF
A POROUS SOLID USING A MIXTURE THEORY†

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Abstract—The thermomechanics of porous solids including the implications of the second law of thermodynamics are studied through application of a mixture theory. This derivation suggests that the pore collapse relation should express the rate of change of either the volume fraction or the distention ratio as an odd function of the difference between the pressure and an equilibrium pressure. An example problem is solved for the dynamic compaction of 78% dense porous aluminum.

1. INTRODUCTION

Porous solids are capable of absorbing large quantities of energy during impact loading. For this reason the behavior of these materials at very high rates of strain is of substantial interest to engineers. Further proof of this can be seen by examining the technical literature on this subject; for example see Refs. [1–6]. Examination of these works will show that they are concerned mainly with questions relating to the mechanics aspects of this problem; i.e. the conservation laws of mass, momentum, and energy. As an example we note the discussions on scaling rules which relate the effective or actual stresses in the solid to the partial stresses which appear in the balance of momentum; see Morland[7] and Garg[8]. Much attention is also paid to formulating pore collapse relations which describe, by means of a constitutive relation, the manner in which the pores close during irreversible crushing of the material.

The word *irreversible* as used in the preceding sentence brings us to the key point of this paper. While investigation of the mechanics aspects of this problem is a difficult and obviously an important part of the total problem, a *complete* treatment of the problem requires an investigation of the implications of the entropy inequality of the second law of thermodynamics. We wish then to present a thermomechanical treatment of the problem.

Two advantages of this approach are, a clearer understanding of the assumptions which result in the usual scaling laws for stress, and an entropy production constraint which applies to the collapse relation. This latter constraint implies that the collapse relation must be stated as a constitutive relation for the material volume fraction rate. This differs from Garg et al.[2] who have used the volume fraction itself rather than its time rate of change to represent the collapse process. Also some resemblance between the rate relation proposed in this paper and the one proposed by Davison[3] is evident; however, Davison's relation is for the material strain and not the volume fraction.

The key to the derivation in this work is to view the porous solid as a mixture consisting of a solid and a vacuum. This viewpoint allows us to draw on the previous work of Truesdell[9], Müller[10], and Bowen and Garcia[11] who gave extensive treatment of the second law of thermodynamics as embodied in the mixture analog of the Clausius–Duhem inequality. The previous work of Garg et al.[2] has some precedence in this instance since they treated a fluid-saturated porous material by means of a mixture theory. Their work is based on the mixture theory of Green and Naghdi[12]; however, Garg et al. do not treat second law questions.

The thermomechanical mixture theory originally suggested by Truesdell contains balance laws of mass, momentum, and energy which have changed little from their original forms. Not surprisingly as pointed out by Truesdell[9] the real debate has been waged about the question of the appropriate application of the second law of thermodynamics.

This debate was mostly resolved with the publication of Müller's paper[10] in which he showed that density gradients must be included in the list of independent or primitive constitutive variables in order to achieve consistency with the second law. In effect he showed that Fick's law which describes long-time atomic level diffusion driven by concentration gradients must be used to remove inconsistencies in the description of entropy production in the mixture.

Superficially this seems to preclude the use of a mixture theory for the purpose of modeling a porous solid. One seems to be faced with the prospect of rationalizing the application of Fick's law to some type of diffusion process in porous solids. However, as we will show this problem can be overcome. In essence the need for the inclusion of density gradients can be removed by introducing the concept of volume fraction.

On a physical level this implies a microscopic separation between the constituents of the mixture. More directly said, the porous solid will be modeled by a theory for an immiscible mixture, whereas Müller's conclusions apply only to miscible mixtures in which microscopic separation does not exist and where volume fraction has no meaning.

Here we should mention that the spirit of this concept is also present in the work of Trapp[13] and Garg et al.[2]; however, since these works did not include a treatment of the second law, the question of density gradients was not discussed. Furthermore Trapp's work was for a purely mechanical system.

In the next section we will present the formulation for a binary mixture which is specialized to immiscible systems. The properties of volume fraction, partial density, and actual density will be defined. In keeping with the assumption of immiscibility the thermodynamic potential functions of each constituent will depend only on the properties of that constituent and not on the properties of the neighboring constituent. We also include a discussion in this section on the need for the thermodynamic potential to depend on the volume fraction if one wishes to describe a mixture with different constituent pressures; that is, a mixture which is not required to obey Dalton's law for partial pressures.

The third section presents a specialization of the mixture theory to the porous solid. This is achieved by allowing the pressure and density of the vacuum constituent to reduce to zero. The interface pressure between the constituents is also zero; however, the volume fraction is obviously not zero. The individual velocities are assumed to be equal so that the pores travel with the solid. The results of this section yield a complete set of balance relations for mass, momentum, and energy and a set of material response functions constrained by the entropy inequality.

In the fourth section an example problem is given which treats the compaction of porous aluminum. A conventional Lagrangian wave propagation code is used to solve the resulting equations, and the numerical results are compared to the transmitted wave profiles from one-dimensional-strain plate-impact experiments on 78% dense porous aluminum reported by Butcher et al.[6].

The paper will conclude with a summary which in part will discuss the importance of the theoretical developments, especially the entropy inequality constraint, in the context of other porous material models.

2. THE MIXTURE THEORY

In this section, the field equations and entropy inequality of the mixture theory proposed by Truesdell, Müller, Bowen and Garcia will be stated in one-dimensional form with the stress in each constituent being described simply as a pressure. As a prelude to this statement several minor points of notation and kinematics must first be clarified.

At every point within the mixture we will speak of average properties of both constituents. Variables such as position, velocity, or entropy will have a subscript; either 1, 2, α, or β. The first two subscripts denote constituents 1 and 2. Whenever subscript α appears, the equation is an abbreviation for two separate equations; the first with $\alpha = 1$, and the second with $\alpha = 2$. Under these conditions when β appears, it acquires a value opposite to that of α.

The reference configurations for each constituent are given by X_α, and the current positions of the material particles originally at X_α are

$$x = \chi_\alpha(X_\alpha, t) \tag{1}$$

where t is the time. The material velocities are

$$V_\alpha = \dot{\chi}_\alpha(X_\alpha, t) \tag{2}$$

where the dot denotes a partial derivative with respect to time.

For an arbitrary function Γ_α, the time derivative following the motion β is denoted as

$$(\dot{\Gamma}_\alpha)_\beta = \dot{\Gamma}_\alpha + V_\beta \Gamma_{\alpha,x} \tag{3}$$

where the subscript $,x$ denotes a partial derivative with respect to position. If we consider the time derivative following the motion of α, then the abbreviated notation will be used such that

$$(\dot{\Gamma}_\alpha)_\alpha \equiv \dot{\Gamma}_\alpha = \dot{\Gamma}_\alpha + V_\alpha \Gamma_{\alpha,x}, \tag{4}$$

The derivation of the field equations for conservation of mass, momentum, and energy follows conventional lines; that is, the change of a given quantity such as mass within a fixed volume is equated to the flow of that quantity across the surface of the volume. For a binary mixture three sets of field equations result representing conservation statements for each of the individual constituents plus also the total mixture.

Application of the principles of conservation of mass, momentum, and energy for each of the constituents result in the following six relations

$$\dot{\rho}_\alpha + \rho_\alpha V_{\alpha,x} = 0 \tag{5}$$

$$\rho_\alpha \dot{V}_\alpha = T_{\alpha,x} + \rho_\alpha b_\alpha + \hat{P}_\alpha \tag{6}$$

$$\rho_\alpha \dot{e}_\alpha = T_\alpha V_{\alpha,x} - q_{\alpha,x} + \rho_\alpha r_\alpha + \epsilon_\alpha \tag{7}$$

where ρ_α is the partial density (as opposed to the actual density); that is, the mass of constituent α in the control volume divided by the total volume occupied by both constituents. b_α is the specific body force. r_α is the specific external heat supply. q_α is the heat flux in constituent α. e_α is the specific internal energy of constituent α. T_α is the driving force in constituent α which as we will see shortly is *not* equal in magnitude to the constituent pressure. Finally, $\hat{P}_\alpha$ and ϵ_α represent momentum and energy production terms associated with the exchange of momentum and energy between the neighboring constituents by means of effects such as diffusion drag and heat conduction. If $\hat{P}_\alpha$ and ϵ_α are zero, these equations assume forms identical to conventional continuum conservation laws. They are decoupled into two separate sets of field equations where each constituent behaves independently. In the most general case we should also have included a mass production term in each of eqns (5) to account for the possibility of mass exchange during chemical reactions; however, this is an extremely complicated and presently unnecessary addition to the formulation, therefore, it will be omitted.

To complete this system of conservation laws, the analogous conservation relations for the total mixture must be included. Truesdell[9] has shown that these conservation relations are equivalent to

$$\hat{P}_1 + \hat{P}_2 = 0 \tag{8}$$

$$\hat{P}_1 V_1 + \hat{P}_2 V_2 + \epsilon_1 + \epsilon_2 = 0 \tag{9}$$

where because chemical reactions have been omitted the mass relation is trivial. Thus the momentum and energy lost by constituent 1 must be gained by constituent 2.

While some of the terms appearing in eqns (5)–(9) must still appear to the reader to be vaguely defined, we hope to clarify them later in the paper.

Besides the material response functions which we will set aside for the moment, there is one remaining law to apply to the mixture, the entropy inequality of the second law of thermodynamics. As stated in the introduction, this is surely the most disputed issue in the field of mixture mechanics. Presently the most popular notion is that only one inequality need be stated and applied to the total mixture. This inequality is then considered as a constraint on the response functions of the constituents; that is, for any and all possible solutions at a point in the

mixture regardless of the associated boundary conditions, the response functions must be chosen so that the entropy inequality is satisfied. The entropy inequality most commonly used to derive these constraints is the mixture at analog of the Clausius–Duhem inequality,

$$\rho_1\dot{\eta}_1 + \rho_2\dot{\eta}_2 + (q_1\Theta_1 + q_2\Theta_2)_x - \rho_1 r_1\Theta_1 - \rho_2 r_2\Theta_2 \geq 0 \tag{10}$$

where η_α is the specific entropy and Θ_α is coldness or inverse of the temperature.

Usually, the system of equations just described is closed with a set of response functions. The major remaining question then concerns the choice of the primitive or independent variables affecting the response of the mixture. This list may include, for example, the partial densities, coldnesses, coldness gradients, and material velocities. As discussed earlier Müller also proposed that the spatial gradients of the partial densities be included in this list. He showed how the constraints imposed by the entropy inequality on miscible mixtures lead to this conclusion. Without inclusion of this variable the properties of one constituent could not depend on the state of deformation in the neighboring constituent.

However, in the present work, we will show that for immiscible mixtures the use of density gradients is not required. We will illustrate this by developing a set of response functions for the immiscible mixture which satisfy the entropy inequality. The list of primitive variables will include the actual densities and not the partial densities, the coldnesses, the coldness gradients, the velocities, and a new set of variables, the volume fractions.

The introduction of volume fractions into the list of primitive variables is crucial to the development of a theory for immiscible mixtures and our special immiscible mixture, the porous solid. To define the volume fractions we must recall the distinction made earlier between the partial and actual densities. If the actual density; that is, the mass of the constituent divided only by the portion of the control volume occupied by that constituent, is denoted as $\bar{\rho}_\alpha$, then the volume fraction is

$$\varphi_\alpha \equiv \rho_\alpha/\bar{\rho}_\alpha \tag{11}$$

where we also have the obvious constraint that

$$\varphi_1 + \varphi_2 = 1. \tag{12}$$

The next step towards the immiscible mixture formulation is the introduction of a special term into the energy production function. This term represents the work done on a constituent by its neighboring constituent when the volume fractions change. Garg et al. use a similar term in their formulation. We will also introduce a term representing energy exchange due to friction or viscous drag as the constituents move relative to each other; i.e. diffuse. Finally, we will also allow an energy exchange by heat conduction. Thus

$$\epsilon_\alpha = -P_0\dot{\varphi}_\alpha + D_\alpha(V_\alpha - V_\beta) + (-1)^\alpha Q \tag{13}$$

where the terms represent, respectively, the volume fraction working, the diffusion drag working, and the interconstituent thermal exchange. The new terms introduced here are: P_0, the interface pressure or the average pressure with which one constituent pushes on its neighbor; D_α, the diffusion drag function equal to the drag force reduced by the percentage of the total drag heating absorbed by constituent α; and Q, the rate or flux of heat from constituents 1 to 2.

At this point eqns (8), (9) and (13) can be solved for $\hat{P}_\alpha$ to obtain

$$\hat{P}_\alpha = (D_\beta - D_\alpha) + P_0\varphi_{\alpha,x}. \tag{14}$$

The second term on the right hand side of (14) is also present in Trapp's work. Equation (14) can be used interchangeably with eqns (8) and (9).

Next our immiscible mixture formulation will require a thermodynamic potential function for each constituent. Since we wish to use densities and coldnesses as primitive variables and since an entropy formulation seems more convenient than an internal energy formulation, the

Massieu potentials Λ_α will be used. They are usually defined by means of the following Legendre transformations:

$$\Lambda_\alpha = \eta_\alpha - e_\alpha\Theta_\alpha. \tag{15}$$

From Müller, Bowen and Garcia it is known that Λ_α cannot be a function of the coldness gradients or the velocities. Also by our definition of an immiscible mixture the functional dependence of Λ_α is limited to the primitive variables of constituent α; that is Λ_1 is a function of $\bar{\rho}_1$, but not of $\bar{\rho}_2$. Thus the functional dependence of Λ_α is limited to

$$\Lambda_\alpha = \Lambda_\alpha(\bar{\rho}_\alpha, \Theta_\alpha, \varphi_\alpha). \tag{16}$$

Obviously, Λ_α differs from a conventional thermodynamic potential since φ_α appears in the list of primitive variables. This is appropriate since eqn (16) must reflect the nature of the mixture.

Here φ_α should be considered a microstructure variable, a variable which represents the local heterogeneous nature of the mixture. In general, convincing arguments can be made for extending the list of variables to include even more microstructural effects. Such a move is in many cases appropriate; however, for the present purposes the volume fractions alone will prove to be sufficient. In the other direction we will show that omission of φ_α from the list would render the resulting theory useless for modeling porous solids.

Familiarity with conventional thermodynamics leads us to expect that the partial derivatives of (16) should be identified with certain physical parameters. Thus as in conventional formulations we define the following relationships:

$$P_\alpha = \frac{\bar{\rho}_\alpha^2}{\Theta_\alpha}\frac{\partial}{\partial\bar{\rho}_\alpha}\Lambda_\alpha \tag{17}$$

$$e_\alpha = -\frac{\partial}{\partial\Theta_\alpha}\Lambda_\alpha \tag{18}$$

where P_α is the local averaged pressure in constituent α. Indeed Müller, Bowen, and Garcia have shown that (18) must always hold. While we have defined the partial of Λ_α with respect to $\bar{\rho}_\alpha$ as the constituent pressure and the partial of Λ_α with respect to Θ_α as the specific internal energy, the partial of Λ_α with respect to φ_α is still undefined. This derivative does not have a counterpart in conventional thermodynamics; however, it does have an important meaning which will be discussed later.

We can now list the complete set of material response functions for the immiscible mixture. They are

$$\Lambda_\alpha = \Lambda_\alpha(\bar{\rho}_\alpha, \Theta_\alpha, \varphi_\alpha) \tag{16}$$

$$P_0 = P_0(\bar{\rho}_1, \bar{\rho}_2, \Theta_1, \Theta_2, \Theta_{1,x}, \Theta_{2,x}, V_1 - V_2, \varphi_1, \varphi_2) \tag{19}$$

$$T_\alpha = T_\alpha(\bar{\rho}_1, \bar{\rho}_2, \text{---} \tag{20}$$

$$q_\alpha = q_\alpha(\text{---} \tag{21}$$

$$D_\alpha = D_\alpha(\text{---} \tag{22}$$

$$Q = Q(\text{---} \tag{23}$$

$$\dot{\varphi}_\alpha = \dot{\varphi}_\alpha(\text{---} \tag{24}$$

where the reasons for casting eqns (24) as rate relations will soon become apparent. Also note that requirements of functional indifference under coordinate frame translation restricts the functional dependence of the response functions to $V_1 - V_2$ and not V_1 and V_2 separately.

As discussed earlier, the entropy inequality (10) places constraints on these relations. For example, we will show that the functional dependence of T_α cannot be as general as that stated in (20). To examine these constraints the chain-rule expansion

$$\dot{\Lambda}_\alpha = \dot{\bar{\rho}}_\alpha \frac{\partial}{\partial \bar{\rho}_\alpha} \Lambda_\alpha + \dot{\Theta}_\alpha \frac{\partial}{\partial \Theta_\alpha} \Lambda_\alpha + \dot{\varphi}_\alpha \frac{\partial}{\partial \varphi_\alpha} \Lambda_\alpha \tag{25}$$

and the identity

$$\dot{\varphi}_\alpha = \frac{\dot{\rho}_\alpha}{\bar{\rho}_\alpha} - \frac{\rho_\alpha}{\bar{\rho}_\alpha^2} \dot{\bar{\rho}}_\alpha \tag{26}$$

obtain from (11) are substituted into (10) to obtain

$$\sum_{\alpha=1}^{2} \left\{ -\frac{\Theta_\alpha}{\rho_\alpha} \dot{\rho}_\alpha (T_\alpha + \varphi_\alpha P_\alpha) + \left[\Theta_\alpha (P_\alpha - P_0) + \rho_\alpha \frac{\partial}{\partial \varphi_\alpha} \Lambda_\alpha \right] \dot{\varphi}_\alpha \right.$$
$$\left. + q_\alpha \Theta_{\alpha,x} + \Theta_\alpha D_\alpha (V_\alpha - V_\beta) + (-1)^\alpha \Theta_\alpha Q \right\} \geq 0 \tag{27}$$

where eqns (5), (7), (13), (17) and (18) have also been used.

According to the arguments of Coleman and Noll[14], it is possible to find a family of admissible solutions to this theory where all terms in (27) save $\dot{\rho}_\alpha$ are constant, and for which $\dot{\rho}_\alpha$ vary between arbitrarily large positive and negative values. Thus the coefficients of $\dot{\rho}_\alpha$ must be zero to ensure that (27) is always satisfied. Therefore

$$T_\alpha = -\varphi_\alpha P_\alpha \tag{28}$$

and the number of independent response functions is reduced by two.

Relations (28) are the scaling rules between the driving pressures in the momentum relations and the actual constituent pressures. They result directly from the assumption that the mixture is immiscible and the assumption that $\dot{\rho}_\alpha$ is not a primitive constitutive variable. The immiscibility assumption implies that the φ_α exist and (16) hold. The omission of $\dot{\rho}_\alpha$ as a primitive variable excludes rate-dependent materials. Thus no reason exists which would substantiate the a priori use of (28) for rate-dependent materials.

The remaining inequality can be satisfied by the sufficient but not necessary conditions that

$$\left(P_\alpha - P_0 + \frac{\rho_\alpha}{\Theta_\alpha} \frac{\partial}{\partial \varphi_\alpha} \Lambda_\alpha \right) \dot{\varphi}_\alpha \geq 0 \tag{29}$$

$$q_\alpha \Theta_{\alpha,x} \geq 0 \tag{30}$$

$$(\Theta_2 - \Theta_1) Q \geq 0 \tag{31}$$

$$D_\alpha (V_\alpha - V_\beta) \geq 0. \tag{32}$$

Relations (29) yield several important results. First the coefficient of $\dot{\varphi}_\alpha$ is not required to be set to zero as was the case with $\dot{\rho}_\alpha$. This is due to the manner in which (24) was used to establish a functional link between $\dot{\varphi}_\alpha$ and the primitive variables. Consequently if $\dot{\varphi}_\alpha$ is omitted as a constitutive function and instead replaced with some other constitutive relation, the coefficient of $\dot{\varphi}_\alpha$ in (29) would have to be zero. Then for example if Λ_α were not a function of φ_α, (29) would require that $P_\alpha = P_0$. All pressures would be required to be equal. Finally (29) suggests the previously mentioned nature of the partial derivative $(\partial/\partial \varphi_\alpha)\Lambda_\alpha$. This will be discussed in complete detail in the next section.

The remaining relations (30)–(32), reveal quite conventional requirements. Firstly, both (30) and (31) require that heat flows from hot to cold. Secondly, (32) requires that the diffusion drag must oppose the relative motion of the constituents.

In the next section to derive the theory for the porous solid many of the complications treated in this section will be eliminated; however, we have retained the generality up to this point to establish one extremely important fact concerning the inclusion of φ_α in the list of primitive variables. To do this we first make the following assumptions:

$$\Lambda_\alpha = -\Theta_\alpha e_\alpha(\bar{\rho}_\alpha, \varphi_\alpha)$$

$$q_\alpha = 0 \qquad Q = 0 \tag{33}$$

$$r_\alpha = 0 \qquad D_\alpha = 0.$$

This in effect reduces the theory to a purely mechanical system. No heat can flow or be stored in the mixture.

By using (5), (13) and (28) the energy eqns (7) become

$$\rho_\alpha \dot{e}_\alpha = \varphi_\alpha P_\alpha \left(\frac{\dot{\rho}_\alpha}{\rho_\alpha} \right) - P_0 \dot{\varphi}_\alpha. \tag{34}$$

We then consider the chain-rule expansion of $\dot{e}_\alpha$ which is

$$\dot{e}_\alpha = \dot{\bar{\rho}}_\alpha \frac{\partial}{\partial \bar{\rho}_\alpha} e_\alpha + \dot{\varphi}_\alpha \frac{\partial}{\partial \varphi_\alpha} e_\alpha \tag{35}$$

where substitution of (17), (26) and (33$_1$) gives

$$\rho_\alpha \dot{e}_\alpha = \varphi_\alpha P_\alpha \left(\frac{\dot{\rho}_\alpha}{\rho_\alpha} \right) - P_\alpha \dot{\varphi}_\alpha + \rho_\alpha \dot{\varphi}_\alpha \frac{\partial}{\partial \varphi_\alpha} e_\alpha. \tag{36}$$

Combination of (34) and (36) finally yields

$$P_\alpha - P_0 = \rho_\alpha \frac{\partial}{\partial \varphi_\alpha} e_\alpha. \tag{37}$$

If we now recall the discussion of the functional dependence of Λ_α, eqn (37) clearly shows the penalty for omitting φ_α from the primitive variables. If φ_α were omitted, the right side of (37) would become zero. Consequently,

$$P_\alpha = P_0$$

or

$$P_1 = P_2 = P_0 \tag{38}$$

that is, both constituents would have to remain at equal pressures. Our first step in deriving the porous solid model will be to set P_0 to zero. This would reduce the theory to a trivial set of equations unless φ_α is retained as a primitive variable in eqns (16). Is interesting to note the parallel between this argument and the discussion immediately following eqns (29).

3. A ONE-CONSTITUENT MIXTURE

The porous solid

The porous solid is considered simply as a mixture of solid and vacuum. To derive the theory for this special case we first require the interface pressure acting between the solid and the vacuum to be set identically to zero. Also all parameters associated with the vacuum except volume fraction and velocity are set to zero. The velocities of the constituents are equal. In addition, we can drop the subscript on the remaining parameters for the solid. Under these conditions the equations of mass, momentum and energy become

$$\dot{\rho} + \rho V_{,x} = 0 \tag{39}$$

$$\rho \dot{V} = -(\varphi P)_{,x} + \rho b \tag{40}$$

$$\rho \dot{e} = -\varphi P V_{,x} - q_{,x} + \rho r. \tag{41}$$

Looking at these equations alone one might at first wonder why we have gone to all this trouble to derive what appears to be the conventional conservation laws of continuum mechanics. Closer inspection, however, reveals several significant differences. Firstly, the product of the pressure and the volume fraction, instead of the pressure alone, appears in (40) and (41). Questions concerning, for example, the placement of φ inside or outside the spatial gradient in (40) are resolved. Secondly, the density terms appearing in these relations are not the actual density of the solid but the partial density which also includes the volume of the pores. Thirdly, these conservation laws are paired with the entropy inequality constraint.

The entropy inequality restrictions become

$$\left(P + \frac{\rho}{\Theta} \frac{\partial}{\partial \varphi} \Lambda \right) \dot{\varphi} \geq 0 \tag{42}$$

$$q \Theta_{,x} \geq 0. \tag{43}$$

Relation (43) still places rather conventional constraints on the heat conduction law; however, relation (42) is a new constraint which holds important implications. To investigate these implications we first restate the necessary response functions from the model. They are

$$\Lambda = \Lambda(\bar{\rho}, \Theta, \varphi) \tag{44}$$

$$q = q(\bar{\rho}, \Theta, \Theta_{,x}, \varphi) \tag{45}$$

$$\dot{\varphi} = \dot{\varphi}(\bar{\rho}, \Theta, \Theta_{,x}, \varphi). \tag{46}$$

Now we can define equilibrium conditions to exist when

$$\dot{\varphi} = q = 0 \qquad \text{(at equilibrium)}. \tag{47}$$

Therefore, (42) and (43) are identically zero at equilibrium. Also the terms on the left side of these inequalities must be at a minimum if the equilibrium condition is to be stable. To obtain the requirements for stable equilibrium, we first consider an equilibrium point defined by $\bar{\rho}_e$, Θ_e, and φ_e. Then if the system is perturbed to some neighboring point $\bar{\rho}$, Θ, φ so that

$$\bar{\rho} = \bar{\rho}_e + \delta \bar{\rho}_0$$

$$\Theta = \Theta_e + \delta \Theta_0 \tag{48}$$

$$\varphi = \varphi_e + \delta \varphi_0,$$

the system will be stable if

$$\left[\left(P + \frac{\rho}{\Theta} \frac{\partial}{\partial \varphi} \Lambda \right) \dot{\varphi} \right]_{,\delta} = 0 \tag{49}$$

$$[q \Theta_{,x}]_{,\delta} = 0. \tag{50}$$

By expanding the derivative of the products in (49) and (50) and by using (47), we arrive at the conditions

$$\left(P + \frac{\rho}{\Theta} \frac{\partial}{\partial \varphi} \Lambda \right) \dot{\varphi}_0 = 0 \tag{51}$$

$$q_{,\delta} \Theta_{,x} = 0. \tag{52}$$

The least restrictive conclusions from (51) and (52) are

$$P = -\frac{\rho}{\Theta}\frac{\partial}{\partial\varphi}\Lambda \tag{53}$$

(at equilibrium)

$$\Theta_{,x} = 0. \tag{54}$$

Again (54) is a conventional requirement; however, (53) defines an equilibrium pressure P_{eq} so that

$$P_{eq} = -\frac{\rho}{\Theta}\frac{\partial}{\partial\varphi}\Lambda. \tag{55}$$

Thus the remaining partial derivative of the thermodynamic potential function has been defined. We now rewrite (42) to obtain

$$(P - P_{eq})\dot{\varphi} \geq 0 \tag{56}$$

which is an extremely important result. We see that the overpressure term, $P-P_{eq}$, in this inequality is zero at equilibrium and thus the entropy inequality is satisfied trivially at equilibrium. Furthermore, away from equilibrium, eqn (56) can only be satisfied if $\dot{\varphi}$ is a response function which is an *odd function of the overpressure* and has the same sign as the overpressure. pressure.

We again emphasize that (56) requires $\dot{\varphi}$ to be the pore collapse relation. One might well suspect that in (24) we dictated $\dot{\varphi}$ to be the collapse relation; however, had we chosen some other dependent variable for (24), it would have been impossible to satisfy (56) unless the overpressure was identically zero for all situations.

4. EXAMPLE PROBLEM

(a) *The response functions for the compaction of porous aluminum*

As a practical illustration of the results derived in the previous sections, a specialized set of response functions will be derived for a porous aluminum. These response functions together with the field equations will be solved numerically and comparisons will be made to experimental data.

We again emphasize that the previous results encompass a general thermodynamic response; however, for the purpose of illustration a thermodynamic potential will be assumed which eliminates many possible thermal effects. The form of the Massieu function we will use is

$$\Lambda = -\Theta E(\bar{\rho}, \varphi) + C(\Theta) \tag{57}$$

which from (17) and (18) gives

$$e = E(\bar{\rho}, \varphi) - C(\Theta)_{,\Theta} \tag{58}$$

$$P = \bar{\rho}^2 \frac{\partial}{\partial\bar{\rho}} E(\bar{\rho}, \varphi). \tag{59}$$

Thus the solid can store heat; however, the pressure is not sensitive to the temperature. From (55) we also have

$$P_{eq} = \bar{\rho}\varphi \frac{\partial}{\partial\varphi} E(\bar{\rho}, \varphi). \tag{60}$$

To ensure that the inequality (56) is identically satisfied for all possible material responses during compaction, we require

$$\dot{\varphi} = \begin{cases} 0; & P < P_{CR} \text{ or } \varphi = 1 \\ K(\bar{\rho}, \theta, \theta_{,x}, \varphi)(P - P_{eq})^{2n+1}; & P \geq P_{CR} \text{ and } 0 < \varphi < 1 \end{cases} \tag{61}$$

where K is a positive definite function, n is a positive integer, and P_{CR} is an arbitrary constant pressure defining the onset of crushing.

We next note that P and P_{eq} are both derived from the same potential function. Thus the relationship we chose for P will affect P_{eq}.

Therefore, if the pressure is given as

$$P(\bar{\rho}) = F\left(\frac{\bar{\rho} - \bar{\rho}^0}{\bar{\rho}}\right) \tag{62}$$

where F is a positive constant and $\bar{\rho}^0$ is the initial density, then (59) can be integrated to yield

$$E(\bar{\rho}, \varphi) = \int \frac{P(\bar{\rho})}{\bar{\rho}^2} \, d\bar{\rho} + h(\varphi) \tag{63}$$

where $h(\varphi)$ is an arbitrary function of φ. Equation (60) then requires that

$$P_{eq} = \bar{\rho}\varphi \frac{\partial}{\partial\varphi} h(\varphi). \tag{64}$$

For completeness and with anticipation of the equilibrium experimental data used to prescribe (64), we assume a form for $h(\varphi)$ so that

$$P_{eq} = \bar{\rho}\varphi\left[k + k_1\left(\frac{1}{1-\varphi}\right) + k_2(1-\varphi)\right] \tag{65}$$

where k, k_1, and k_2 are constants. It is seen that this choice for P_{eq} has a singularity at $\varphi = 1$. We could have chosen a form without such a singularity. The form used in (65) simply reflects this specific materials resistance to being fully compacted, and it does not restrict application of the general information to fully compacted materials.

To illustrate another point we again use the chain-rule expansion for $\dot{e}$ in the energy balance relation (41) to obtain

$$\rho \frac{\partial e}{\partial \Theta} \dot{\Theta} = (P - P_{eq})\dot{\varphi}. \tag{66}$$

Since e increases with decreasing coldness, $(\partial e/\partial\Theta)$ is negative. Thus from (56)

$$\dot{\Theta} \leq 0 \tag{67}$$

or the temperature only increases during crushing. Therefore in this particular model changing the volume fraction; that is, crushing the pores, generates heat which cannot be recovered.

(b) *Evaluation of the response function constants*

The paper on the compaction of porous aluminum by Butcher, Carroll and Holt[6] contains data on 78% dense material subjected to impact with a flat flyer plate. The configurations of these experiments were chosen to sustain a condition of one-dimensional strain. The impact produced a sharp step loading on the specimens even when measured on submicrosecond time scales. During and after passage of the initial disturbance through the solid, data was collected as the specimen relaxed to an equilibrium state resulting from the sustained loading of the step wave.

One of the experimental configurations used in their paper is illustrated in Fig. 1. During the short time in which data are collected the planar impact produces a one-dimensional-strain stress wave along the centerline of the configuration. The disturbance propagates to the circular mirror located at the exit buffer-window interface, and the ensuing motion of the mirror is monitored with an optical interferometer.

It was observed in these experiments that the porous solid sustained a precursor wave of
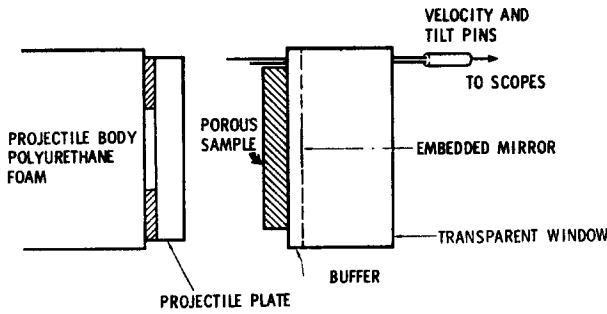
Fig. 1. Schematic of flyer-plate experiment.

amplitude 0.8×10^8 Pa and traveling at 4.11 km/s. If we assume that the volume fraction did not change under the influence of this precursor wave, then φ can be uniformly divided out of eqns (39)–(41) in which case it is easily shown that

$$F = \bar{\rho}^0 C_P^2 \tag{68}$$

where C_P represents the precursor wave speed. Since the initial actual density of the aluminum equals 2.77 Mg/m^3, the corresponding value for F is 467.9×10^8 Pa. Also since crushing occurs after passage of this precursor, P_{CR} equals the amplitude of this wave divided by the initial volume fraction, $P_{CR} = 1.025 \times 10^8$ Pa.

The precursor wave is followed by the larger, slower compaction wave which crushes the pores. Data relating the equilibrium pressure attained behind the compaction wave to the final volume fraction are summarized in Fig. 2. Here the inverse of the volume fraction or the distension ratio is plotted against pressure. In the literature on this subject the distension ratio is the more common term, and it is usually denoted by α.

Equation (65) must be fit to the data in Fig. 2. The fit attained with $k = 1.73 \times 10^8$ Pa; $k_1 = 0.0602 \times 10^8$; and $k_2 = -7.00 \times 10^8$ Pa is shown by the solid curve.

The remaining response function to be evaluated is the collapse relation (61). An obvious simplification to this relation is to assume that the function K is a positive constant. By trying various constant values of K and n in the equations, we can see that neither K nor n seems to affect the speeds or amplitudes of either the precursor or compaction wave. Only the rise time of the compaction waves appears to be affected. Furthermore, if n is zero the rise time of the compaction wave is independent of the amplitude, and if n is one the rise time of the compaction wave decreases as the amplitude increases.

Further reflection on the problem suggests that the average pore dimension in the actual
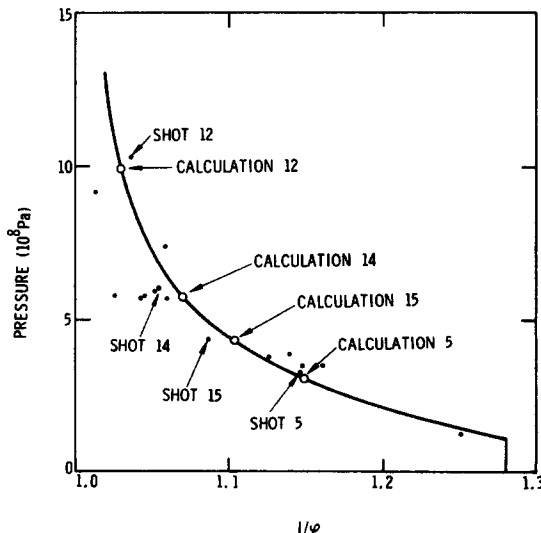

Fig. 2. Equilibrium data and theoretical curve used in calculations.

material should have a great influence on the selection of K and n. In fact this dimension when divided by the average particle velocity of the compaction wave was used to obtain an estimate of the magnitude of $\dot{\varphi}$ and a quite reasonable initial estimate of the values for K and n. Ultimately the values of K and n were determined by empirical fit to the experimental curves. A final value of $n = 1$ was chosen to account for the general steepening of the compaction wave with amplitude. A value of $K = 2.5 \times 10^{-19}/\text{Pa}^3\text{s}$ was then determined by fit to shot 15.

(c) *Numerical calculations and comparison to experiment*

Several calculations were performed for the experimental configuration shown in Fig. 1. The calculations were carried out with WONDY IV, a Lagrangian wave propagation code based on explicit finite differencing[15]. Because of the internal structure of this code and the similarity of the mixture conservation equations to conventional wave code equations, incorporation was quite direct. The only unusual feature of this procedure required that values of φ be advanced in time by using an auxiliary integration routine to integrate relation (61). To do this the new value of $\bar{\rho}$ was first computed and intermediate values of $\bar{\rho}$ were evaluated by interpolation between old and new values of $\bar{\rho}$. This in effect reduced (61) to an ordinary differential equation in φ between each successive time step.

The specific experiments calculated by this procedure are listed in[6] as shots 5, 12, 14 and 15. These experiments are also explicitly labelled in Fig. 2. Information on the critical parameters of each experiment is listed in Table 1.

The transmitted wave profiles obtained from the calculations are compared with the experimental data in Figs. 3–6. The calculations and experimental records show the same general behavior. A fast moving precursor of constant amplitude leads the record. This wave raises the pressure in the solid to the yield point where crushing begins; however, no crushing occurs until the arrival of the slower moving compaction wave. The speed of the compaction wave depends on the magnitude of the applied load. Here the volume fraction increases smoothly from its initial value of 0.78 to its final value which is shown in Fig. 2.

As to the comparisons between the calculations and the experiments, the agreement is good in every case except shot 12 in Fig. 6. In Figs. 3–5 the precursor and compaction waves are in good agreement for both speed and rise time. In Figs. 4 and 5 the amplitude of the slower compaction wave is slightly underestimated.

Table 1.

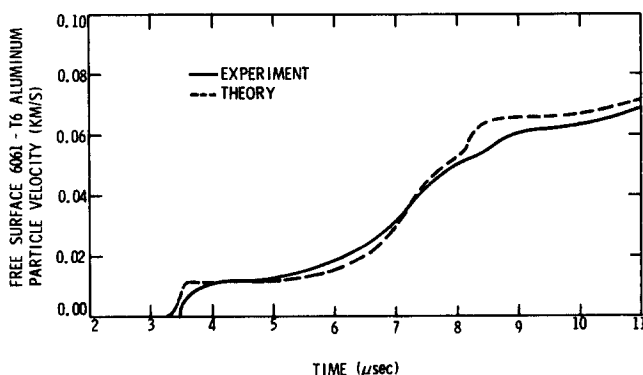| Shot No.	Flyer Plate		Sample	Buffer		Window
	Material	Velocity (km/s)	Thickness (mm)	Material	Thickness (mm)	Material
5	6061-T6 Al	0.1262	4.47	6061-T6 Al	15.733	None
12	6061-T6 Al	0.36578	7.0535	Fused Quartz	3.2563	Fused Quartz
14	6061-T6 Al	0.24444	4.572	Fused Quartz	6.4364	Fused Quartz
15	6061-T6 Al	0.18075	4.506	Fused Quartz	6.4313	Fused Quartz

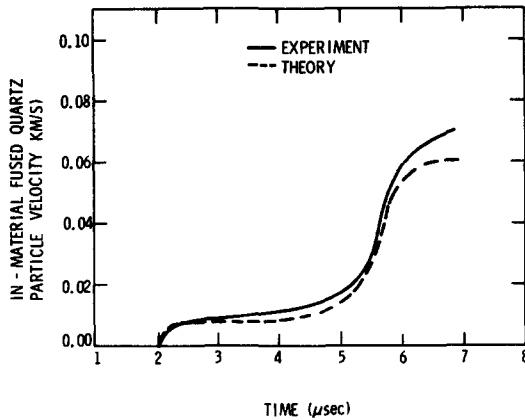Fig. 3. Comparison of theory to data from shot 5.

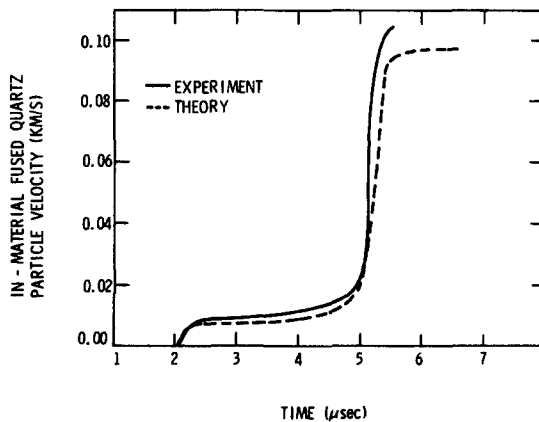Fig. 4. Comparison of theory to data from shot 15.

Fig. 5. Comparison of theory to data from shot 14.

In Fig. 6 the calculation differs from the experimental data in several respects. Firstly, while the calculation predicts the same precursor speed as in the other calculations and experiments, the experimental precursor speed is somewhat slower. Secondly, the breakaway at the toe of the compaction wave in the experimental record shows an extra hump which is not present in the other records. Thirdly, a spike or possibly an oscillation is present at the head of the compaction wave. No explanation is available for these phenomena, and the current form of the mixture model apparently cannot account for this behavior. Unfortunately, even if the theory did predict the spike, because of the nature of the wave code, the numerical algorithm would tend to erase it.

An additional set of calculations were run for the purpose of illustrating the trend in the results as the applied loading is increased. This time the thickness of the specimens for all practical purposes were made infinitely large. The particle-velocity histories in Fig. 7 were obtained from a material point originally located 5 mm beyond the impact plane in the porous sample. This figure clearly shows that the precursor amplitude and speed are constant from calculation to calculation; however, the speed and rise time of the compaction wave change.

5. CONCLUSIONS

By specialization of Truesdell, Müller and Bowen's mixture work to immiscible mixtures, we have presented a formulation for porous solids which includes not only a complete discussion of the conservation laws for mass, momentum, and energy, but also extensive treatment of the entropy inequality and the constraint that it places on the response functions. We have shown how the assumptions of immiscibility and material rate independence lead to
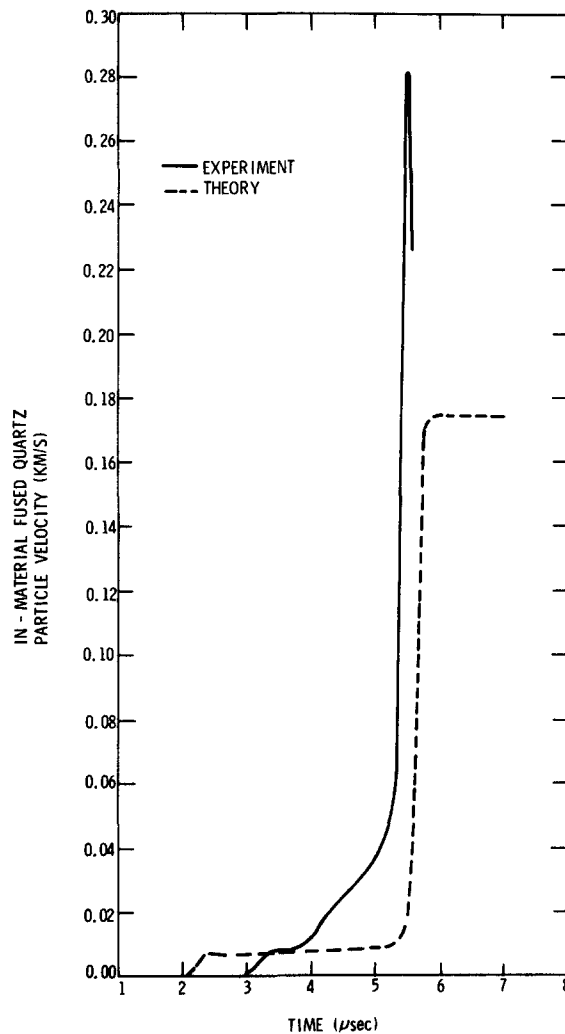
Fig. 6. Comparison of theory to data from shot 12.

the pressure scaling rules (28), and how the entropy inequality requires a pore collapse relation wherein $\dot{\varphi}$ is expressed as an odd function of the overpressure.

A more subtle result in this work is that the response functions for P and P_{eq} cannot be independently chosen. They are both derived from the same thermodynamic potential function. Thus, for example, once eqn (62) was determined in the sample formulation then eqn (64) was fixed, and while $h(\varphi)$ was still arbitrary the product $\bar{\rho}\varphi$ was not.

We will close with several comments on other work referenced herein. These comments relate mainly to the pore collapse relations and their compatibility with the entropy inequality (56).

The P-α model of Herrmann[1] is a special case of this work. The distension ratio, α, is the inverse of the volume fraction. Herrmann's additional assumption is that $P = P_{eq}$ under all conditions. Thus (56) is always satisfied trivially. A pore collapse relation is not needed.

The fluid-saturated-porous-media model of Garg et al.[2] should also have some correspondence to this work when the fluid is not present. In this case φ is the response function representing pore collapse and not $\dot{\varphi}$. Thus (56) is not explicitly satisfied.

In Butcher's work[6] a rate relation for the distension ratio is prescribed in a manner similar to relation (61). We note that this relation also depends on the difference between the local pressure and an equilibrium pressure; however, an additional compliance relating an elastic change in volume fraction to a change in pressure is also included. Thermodynamical constraints are not considered in this work.

In Carroll and Holt's model[4] the pore collapse mechanism is investigated through the
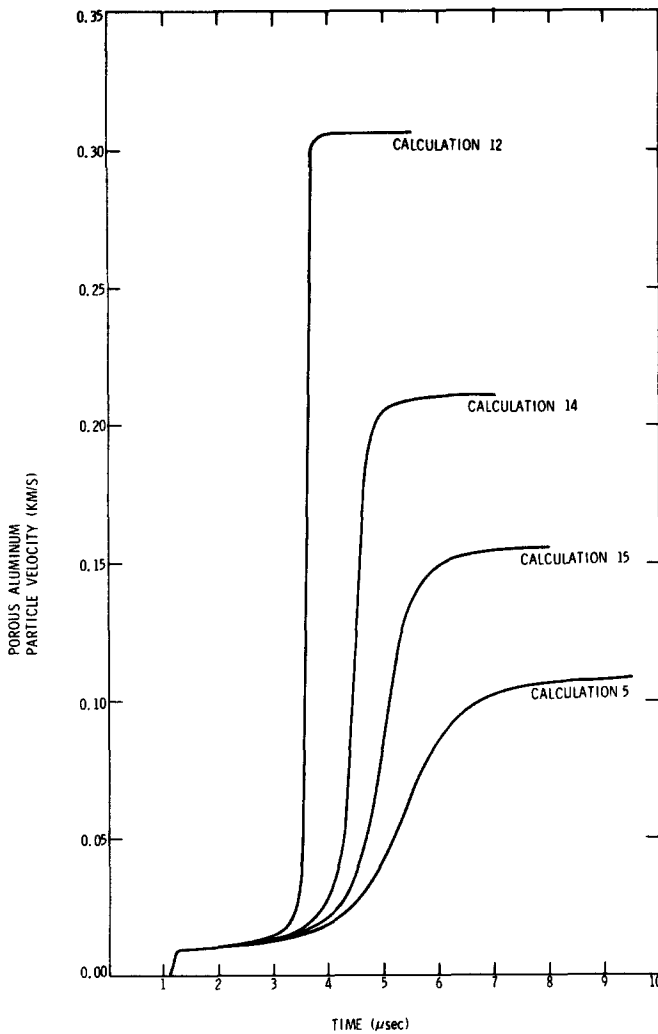
Fig. 7. Comparisons of theory for in-material response at propagation distance of 5 mm.

study of a hollow elastic–plastic sphere. This model which attempts to treat complex local inertial effects yields a collapse relation of the type

$$D(\varphi) = P - P_{eq} \tag{69}$$

where D is a second-order differential operator. Here (69) replaces (61). In this case a direct comparison is difficult since localized inertial effects would first have to be introduced into our mixture formulation. Some attempts have been made in this direction as evidenced by the works of Passman[16] and Goodman and Cowin[17]; however, this is beyond the scope of the present work.

Finally in Davison's work[3] a relation similar to (61) is considered; however, it is written for strain rate and not volume fraction rate. In general one might expect both the strain and volume fraction rates to have quite similar behavior, but again this relation does not explicitly satisfy (56).

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